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BPS spectrum of 5 dimensional field theories, (p,q) webs and curve counting

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Abstract: We study the BPS spectrum of supersymmetric 5 dimensional field theories and their representations as string webs. It is found that a state of given charges exists when it has a representation as an irreducible string web. Its spin is determined by the string web. The number of fermionic zero modes is $8g + 4b$, where g is the number of internal faces and b is the number of boundaries. In the lift to M theory of 4d field theories such states are described by membranes ending on the 5-brane, breaking SUSY from 8 to 4, and g becomes the genus of the membrane. Mathematically, we obtain a diagrammatic method to find the spectrum of curves on a toric complex surface, and the number of their moduli.

Keywords: Field Theories in Higher Dimensions, Solitons Monopoles and Instantons, M-Theory, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

In [1] (p, q) webs of 5-branes were shown to serve as a model for (some) 5d supersymmetric field theories. The definitions and relations are reviewed in section 2. In that paper some open questions were mentioned, and the first two were:

1. 5d field theories can be constructed both from brane configurations (webs) [2, 3] and by compactifying M theory on a Calabi-Yau (CY) threefold while decoupling gravity by shrinking a 4 cycle. It was conjectured that there is a mapping between webs and Calabi-Yau manifolds.

2. The BPS spectrum was shown to be encoded as (p, q) string webs inside the 5-brane web. It was not known how to determine the multiplet type (hyper,vector..) of a state, and how to determine whether a marginally bound state exists.

The first question was answered by Leung and Vafa [4] showing how the (p, q) web is related to toric manifolds. We review that connection in section 3. In this paper we address the second question and study the BPS spectrum. Consider a string web that has boundaries on a 5-brane web. One has to find the zero modes, and look for the ground states of the quantum mechanics which they define. The transformation properties of these ground states under the super-Poincare group give the multiplet type of the particle.

We find that

1. The ground state of the string web is indeed supersymmetric.

2. A marginally bound state exists whenever the string web is irreducible, as anticipated in [1], namely when a generic web cannot be decomposed into sub-webs. Thus we find sites on the charge lattice which are occupied by marginally bound states, but it is not clear whether, in general, other sites are occupied.

3. The multiplet type of a string web with no internal faces (corresponding to a zero genus curve) is determined. Recall that a massive BPS state in 5d transforms under $SO(4) = SU(2)_p \times SU(2)_b$, where the subscripts p, b refer to preserved and broken supersymmetries. The particle transforms as $(j)_p \otimes H_0$, where $H_0 = 2(0)_b \oplus (1/2)_b$ is a half hypermultiplet, generated by the broken supersymmetries. The quantization of a web might result in a number of such representations. The highest spin representation, j, is

$$2j = n_X - 1\,, \tag{1.1}$$

   where n_X is the number of external legs. $j = 0$ is a hyper and $j = 1/2$ is a vector.

4. For a general web we find the zero modes, though presently we do not solve the quantum mechanics. The number of bosonic zero modes (ignoring the 4 translation modes) is $n_{BZM} = n_X + F_{int} - 1 = F - 1$, where F_{int} in the number of internal faces, and F is the total number of faces. In case there are "hidden" faces, F should be replaced by the total number of points in the respective grid diagram [1]. Comparison with the geometrical analysis shows that these moduli are complexfied. The bosonic zero modes define the moduli space for the web, which is the target for the quantum mechanics. The number of fermionic zero modes [5], and hence the maximum spin in a reduction to 4d, j_4, are

$$\begin{aligned} n_{FZM} &= 8F_{int} + 4n_X\,, \\ |j_4| &\leq n_X/2 + F_{int}\,. \end{aligned} \tag{1.2}$$

   After lifting to M theory we find the number of fermionic zero modes for a massive membrane ending on a 5-brane and breaking 8 supercharges to 4

$$n_{FZM} = 8g + 4b \tag{1.3}$$

   generalizing the results of [6, 7, 8], where $b = n_X$ is the number of boundaries and $g = F_{int}$ is the genus.

The method is to use [4] to translate webs into geometry, and then borrow the results of Witten [9]. The number of fermionic zero modes is found by solving the zero modes equation, or by operating with locally preserved supersymmetries on the bosonic zero modes [5]. The argument is presented in section 4. Three examples are worked out in section 5.

Mathematically, we count curves [10, 11] on a toric complex surface (4 real dimensions), or a few intersecting toric surfaces. We find a diagrammatic method to determine which homology classes are occupied by curves, and the number of their moduli.

We would like to point out directions for further research:

- We determined so far the zero modes but did not solve for the ground states of the quantum mechanics. The moduli space contains singularities, in which webs or curves are reducible, which may affect the quantum mechanics. In particular, for the zero genus case we know the highest spin multiplet, but others might arise at the singularities.
- It is known that irreducibility is a sufficient condition for a bound state. What is the necessary condition?
- It was shown that webs can describe toric geometries. It is still unknown whether there are brane configurations for other geometries, such as $\mathbf{B}_k$ for $3 < k \leq 8$.

2. Review of (p, q) webs

Five dimensional field theories were thought to be non-existent since they are not renormalizable. However, 5d conformal theories enable the definition of UV fixed points, which can be perturbed in the IR, for example by Yang-Mills terms. The study of 5d, supersymmetric $N = 1$ (8 supercharges) field theories was initiated by Seiberg [14] and developed in [15, 16, 17, 18, 19]. These papers took both a field theoretic approach and a geometric approach considering M theory on a CY. It was found that the theory is characterized by a prepotential which is cubic in the moduli. The first derivatives give the tension of BPS monopolic strings and the second derivatives give the running coupling constant (the metric on moduli space). There exists a quantized parameter with a finite number of allowed values, called c, which is related to the coefficient of the Chern-Simons term.

In [1] (p, q) webs of 5-branes were shown to serve as a model for (some) 5d supersymmetric field theories. A (p, q) *web* is defined by a collection of edges (finite or semi-infinite) in the (x, y) plane. Each edge is carrying a (p, q) label, where (p, q) are relatively prime integers. The slopes are constrained according to

$$\Delta x : \Delta y = p : q \,. \tag{2.1}$$

The edges are allowed to meet at vertices provided that the (p, q) charge is conserved

$$\sum_i \begin{bmatrix} p \\ q \end{bmatrix}_i = 0 \,. \tag{2.2}$$

The slope condition (2.1) allows a sign ambiguity for (p,q) , which is fixed by a choice of orientation. To check the vertex condition (2.2) the edges should be oriented to be all incoming, or all outgoing. Assigning to a (p,q) edge a tension

$$T_{p,q} = \sqrt{p^2+q^2}\,, \tag{2.3}$$

these conditions ensure the equilibrium of forces at a vertex. See figures (1,2) and others for examples. The definition of the web [3, 1] developed gradually. It originated with the work of Schwarz [20] on (p,q) strings. The vertex condition was found by Aharony, Sonnenschein and Yankielowicz [21]. The BPS nature of a vertex at mechanical equilibrium was discussed by Schwarz [22], and finally the full definition including the slope condition was given by Aharony and Hanany [3]. A proof that a string web preserves 1/4 of the supercharges was given in [23, 13].

String theory has two objects that can realize a (p,q) web: the (p,q) strings and the (p,q) 5-branes of type IIB. Employing the (p,q) 5-branes we can take the other $4+1$ dimensions of the 5-brane to be common, and get as the low energy theory of the web a 5d $N=1$ field theory [3]. In accordance with the general construction of brane configurations [2], this requires the energies on the brane to be much smaller than the Planck mass so that gravity decouples. For a general complex scalar of IIB, $\tau = \chi/2\pi + i/\lambda$, where χ is the RR scalar and λ is the IIB string coupling, the plane of the web would be related to the physical plane of the 5-brane through a linear transformation. In [1] it was conjectured that τ is a redundant parameter for the low energy field theory and can be set to the S self-dual point $\tau = i$, so we can identify the two planes. It was found that for many field theories (at least for those that are related to toric spaces) the vacuum structure could be read from the web [1]:

Field theory	Web
• Local $U(1)$ gauge symmetries	Local deformations of the web
• Global U(1) symmetries	Global deformations
• Tension of monopolic string	Area of face
• Running coupling constant	Mass of a local deformation mode (roughly the perimeter of the face)

These quantities allow to determine the prepotential of the theory.

The BPS spectrum was found to be described in terms of strings ending on the 5-brane [1]. In addition to the strings, *"strips"* or strings inside the 5-brane are a second building block. Strips differ from strings by their tension, which is (for $\tau = i$)

$$T^{strip}_{p,q} = 1/\sqrt{(p^2+q^2)}\,. \tag{2.4}$$

Webs of strings and strips can be built according to the following rules:

- String boundary: a (p,q) string can only end on a (p,q) 5-brane.
- (String) slope: $\Delta x : \Delta y = -q : p$. This assures there is *no parallel force* to the 5-brane.
- String vertex. The (p,q) charge is conserved at string vertices.
- Strip type. A (p,q) strip lies inside a (p,q) 5-brane.
- String-strip vertex - *"bend"*. Strips are allowed to bend out of the 5-brane to become strings. Given an integer vector (p,q) , there is an integer vector (e,f) that completes it to a basis of $\mathbb{Z}^2$. A collection of a (p,q) strips are allowed to bend into a $a(e,f)+b(p,q)$ string for arbitrary b. Note that the no-parallel-force rule is satisfied. If a,b are not relatively prime this configuration is reducible to a multiple of bend configurations.
- Strip boundary. Strips end at vertices of the 5-brane web, but they cannot end on any vertex. A strip should be thought as ending on one of the other 5-brane edges coming out of a vertex. The sum of all (p,q) charges of boundaries is required to vanish. For example, a strip is allowed along a 5-brane edge iff there are parallel and opposite 5-branes coming out of the two vertices. This would be a hyper.

Examples of string webs are shown in figures 4, 5, 6, 8.

In M theory these configurations are known to be lifted to membranes ending on the M5-brane. The membranes are holomorphic in a complex structure that is orthogonal to the 5-brane complex structure.

At this point we can give a specific example for the question that is discussed in this paper. Consider the pure $SU(2)$ theory (see figure 6a). It has two simple BPS states. The W, represented by the vertical line, has charge $(n_e, I) = (1,0)$, where n_e is the electric charge, and I is the instanton charge. The instanton, represented by the horizontal line has charge $(1,1)$. The type of question we would like to answer in this paper is whether a state of charge $(n_e, I) = (2,1)$ exists (see figure 6b), and if there is one we would like to determine its spin. Since the BPS algebra is real such a state would be a marginally bound state of the W and the instanton.

In addition to describing the prepotential and the BPS spectrum, the web provides a picture of the different singularities in moduli space. A flop transition is shown in figure 1. The mass of the hyper is proportional to the finite segment. The transition point (figure 1b) has two intersecting lines creating a 4 junction. Note that this transition between the two Kahler cones is natural in the web. Figure 2 shows a singularity where a vector becomes massless enhancing the local gauge symmetry as two parallel 5-branes coincide. Such a singularity is located on the boundary of the extended Kahler cone.

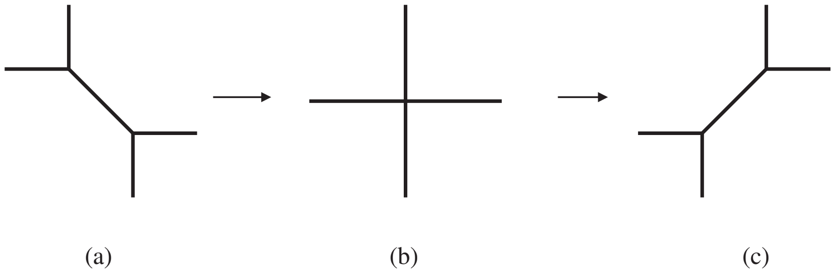

Figure 1: A flop transition.

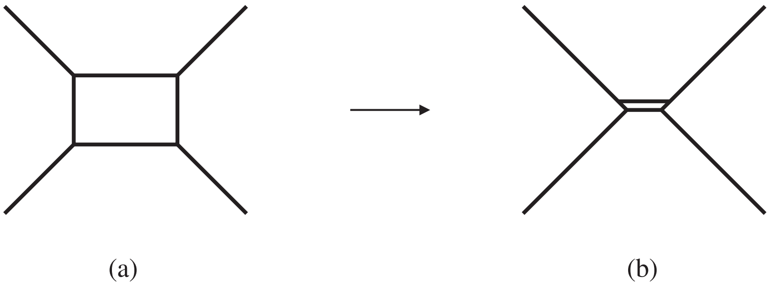

Figure 2: An enhanced gauge symmetry singularity.

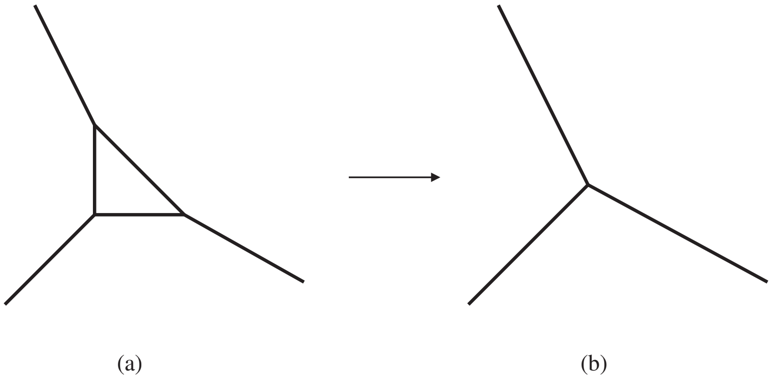

Figure 3: A conformal theory singularity.

Figure 3 shows a face shrinking to zero size. Here a whole tower of BPS states becomes massless and we obtain a conformal theory. This singularity is again located at the boundary of the extended Kahler cone. These phase transitions and their relation to black hole physics were also studied in [24].

Webs supply the Seiberg-Witten curve for a compactified 5d theory [25, 26] through a lift to M theory, in the same geometrical way that was found in 4d [27]. The curve can be read from the corresponding grid diagram. Integrable systems give another approach to 5d field theories [28, 29, 30, 31]. There, one considers formally the "area" differential $dx \wedge dy = d\lambda_{SW}$ to be the fundamental two form of a dynamical system $dp \wedge dq$, except that the area differential is complex rather than real.

3. Webs and toric geometry

Let us describe the correspondence between webs and Calabi-Yau spaces with a shrinking 4-cycle [4].

Web	Calabi-Yau
• Web	Moment map

It was found that webs correspond to spaces, X that have a representation as a toric variety. A d complex dimensional toric variety is acted upon by the algebraic torus, $\mathbb{C}^{*d}, \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. It is defined by a fan $\Delta \subset N$ where N is a d (real) dimensional vector space, and Δ is collection of cones that constitutes the required combinatorial data [32]. N is the space of the grid diagram, and it seems that any grid diagram can be mapped to a fan. The dual space to N is called M. The web is the image of the "moment map" $X \to M$.

• Compact faces	The shrinking 4-cycle.

The 4-cycle is built from the planar face by "fibering" a torus, T^2 above each point.

• A (p,q) edge	Locus of degeneration of a (p,q) cycle of the torus.

This "fibration" has singularities where a (p,q) cycle on the torus shrinks. These are the (p,q) edges in the web.

• External legs	The normal bundle to the 4-cycle.

To give a local description of the CY near a 4-cycle we still need to know the structure of its normal bundle. Since the CY has $d = 3$ the compact faces that have T^2 "fibered" over them, are already surfaces of degeneration of a cycle in T^3. So on each face there is a cycle of T^3 that degenerates, and on each edge actually two cycles degenerate.

•The vertex condition	The CY condition $c_1 = 0$
• Triple intersections of faces/edges	The CY triple intersection form on 4-cycles/2-cycles.

In addition, the vertex condition, and the intersections of the web have CY parallels. The Seiberg - Witten curve, mentioned at the end of the previous paragraph, seems to be related to the CY through mirror symmetry: it was found that the mirror to a CY with a shrinking 4-cycle includes a Riemann surface which depends only on the local 4-cycle and not on the surrounding CY [33].

4. BPS spectrum and curves

Having the mapping from the web to the Calabi-Yau space, we can use the results of [9] to determine the BPS spectrum.

Web	Calabi-Yau
• String web	Curve

In the CY picture a BPS state comes from a membrane wrapping a curve. The moment map will send this curve to a string web (assuming the curve is toric as well). A marginally bound state exists when the curve is irreducible [9]. This corresponds to the string web being irreducible.

• Irreducible string web	Irreducible curve

For *zero genus curves* the highest spin, j, of the BPS state is determined by the dimension of its moduli space $\mathcal{M}$ [9]

$$j = dim_{\mathbb{C}}\mathcal{M}/2\,. \tag{4.1}$$

Let us recall the derivation of this result. A massive BPS state in 5d transforms under $SO(4) = SU(2)_p \times SU(2)_b$. Under this group the 8 supercharges transform as $2(\mathbf{\frac{1}{2}})_p \oplus 2(\mathbf{\frac{1}{2}})_b$. The BPS state preserves 4 of the 8 supercharges. Without loss of generality we can take them to be $2(\mathbf{\frac{1}{2}})_p$, while the broken ones are $2(\mathbf{\frac{1}{2}})_b$. The 4 broken supercharges act on the state to produce half a hypermultiplet $H_0 = 2(\mathbf{0})_b \oplus (\mathbf{1/2})_b$. To get the total representation we have to quantize the quantum mechanics on the moduli space of the curve and tensor it with H_0.

The supersymmetric quantum mechanics takes place on the moduli space $\mathcal{M}$ which is a Kahler manifold. The Hilbert space is the collection of (p, q) differential forms on $\mathcal{M}$, and the 4 supercharges are represented by $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*$. For zero genus curves the fermions are simply in the tangent bundle of the moduli space (as can be seen from the zero mode counting). Supersymmetric ground states exist - they are the harmonic forms. These states constitute a representation of $SU(2)_p$ in the following way: denote by J_3, J_+, J_- the standard $SU(2)$ generators. J_3 acts by multiplying a (p, q) harmonic form by $(p + q - dim_{\mathbb{C}}\mathcal{M})/2$, while J_+, J_- act by a wedge product or contraction with the Kahler form. The harmonic forms split into a number of $SU(2)$ representations, the highest of which is (4.1).

To map this discussion to the webs we recall that the image of a curve is a string web. Every complex modulus of the curve is a real modulus, or a deformation mode, since all phases are lost in the transition.

• Deformation mode	Complex modulus

The number of deformations, or bosonic zero modes, of a finite string web, n_{BZM} is given by n_{BZM} = #(grid-points)-1 (ignoring the 4 translations), where we use the number of points in the corresponding grid diagram, including boundary points. In terms of the web $n_{BZM} = F_{int} + n_X - 1$ where F_{int} is the number of (possible) internal faces, and n_X is the number of external legs [1]. To see that imagine that the external legs were infinite and add up the local modes $n_L = F_{int}$, the global modes $n_G = n_X - 3$ (which are also local for finite external legs) and 2 translation modes.

When we consider non-zero genus curves, there are more fermions, and the bundle of fermions is larger than the tangent bundle. To find the fermionic zero modes of webs one should solve the equations given in [5]. Alternatively, a short cut can be used. There are $n_X + F_{int} - 1$ bosonic zero modes, plus 4 space time translations. Whereas the "supports" of $n_X - 1$ modes are preserved only by the 4 globally preserved supersymmetries, the "supports" of the F_{int} modes are preserved by 8 supersymmetries. The reason is that these modes are supported on internal faces, and do not include the 5-brane boundary. Finally we should add the 4 zero modes which are the partners of space time translations. Summing all the contributions we find the number of fermionic zero modes to be

$$n_{FZM} = 8F_{int} + 4n_X \,. \tag{4.2}$$

Lifting to M theory, the string web turns into an open membrane ending on a 5-brane, and the 5d theory acquires a compact radius. Since the genus of the smooth membrane is $g = F_{int}$ and the number of holes is $b = n_X$, we get

$$\begin{aligned} n_{BZM} &= 2(g + b - 1) \\ n_{FZM} &= 8g + 4b \,. \end{aligned} \tag{4.3}$$

This generalizes the results of [6, 7] that found that a disk shaped membrane results in a hyper and a cylinder results in a vector.

5. Examples

5.1. E_0 or $\mathbf{P}^2$

Consider the E_0 theory. The relevant web is shown in figure 4a. The theory has a single simple BPS state, which we denote by Δ, shown as the dashed line. Consider a state made out of two Δ's (figure 5a). Although this string web is reducible, a general deformation of it is irreducible (figure 5b), so there is a marginally bound state. Generalizing to n Δ particles (4b) we see that a bound state exists, and counting deformations

$$n_{BZM} + 1 = F = (n+2)(n+1)/2 \,. \tag{5.1}$$

In particular, since the Δ does not have internal faces, we find that Δ *has* $j = 1$. Note that in general the number of fermionic zero modes of a bound state is higher than the sum of contributions from the individual constituents.

The corresponding toric geometry is a shrinking $\mathbf{P}^2$. The same computation can be repeated in elementary geometrical methods. Δ correspond to a line, $\mathbf{P}^1$. A bound

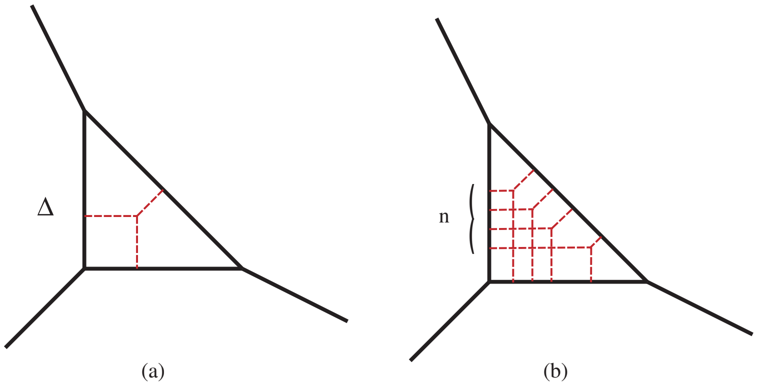

Figure 4: The E_0 theory. (a) The BPS state Δ. (b) A bound state of n Δ's.

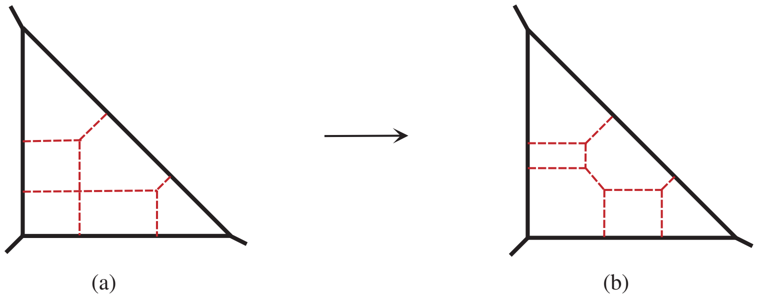

Figure 5: The BPS state made out of 2 Δ's. (a) A reducible configuration. (b) A general irreducible configuration.

state of n Δ's is described by a homogeneous polynomial of degree n in the homogeneous coordinates of $\mathbf{P}^2$. Counting the number of coefficients for such a polynomial we find that

$$\dim_{\mathbb{C}}\mathcal{M} + 1 = (n+2)(n+1)/2\,. \tag{5.2}$$

Comparing it with the previous result (5.1) we verify that $n_{BZM} = \dim_{\mathbb{C}}\mathcal{M}$.

5.2. E_1 or $\mathbf{P}^1 \times \mathbf{P}^1$

Consider the theory of pure $SU(2)$ gauge symmetry, a deformation of the E_1 fixed point (figure 6). This theory was already mentioned at the end of section 2. Consider a state made out of m W's and n instantons (figure 6b). A bound state exists whenever $m, n > 0$. The sites which are certainly occupied on the (n_e, I) charge lattice are shown in figure 7. The number of bosonic zero modes is

$$n_{BZM} + 1 = F = (m+1)(n+1)\,. \tag{5.3}$$

In particular the bound state of the W and the instanton (figure 6a) has $j = 3/2$.

The corresponding toric 4 cycle is $\mathbf{P}^1 \times \mathbf{P}^1$. The curve spectrum on it consists of bihomogeneous curves of bidegree (m, n). They are indeed irreducible when $m, n > 0$, and the dimension of the moduli space is $\dim_{\mathbb{C}}\mathcal{M} + 1 = (m+1)(n+1)$.

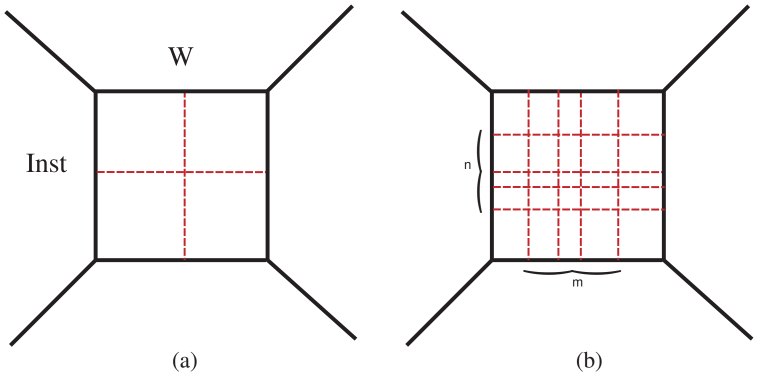

Figure 6: The E_1 theory. (a) The W and the instanton. (b) A bound state of m W's and n instantons.

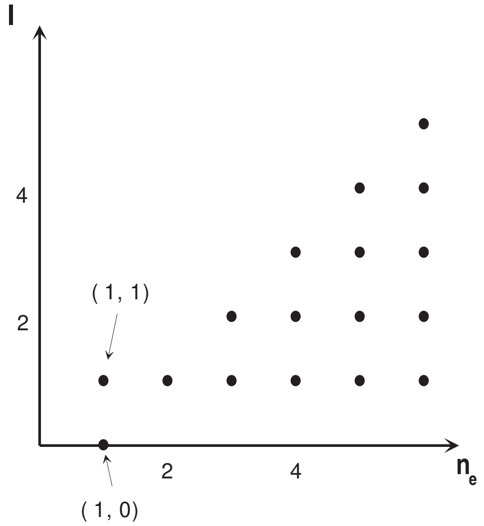

Figure 7: The sites on the E_1 charge lattice which are occupied by BPS states. n_e is the electric charge, I is the instanton number.

5.3. $\tilde{E}_1$ or $\mathbf{F}_1$

Next we consider the theory of pure $SU(2)$ with a theta angle $\theta/\pi = 1$ (mod 2) (figure 8). It is a deformation of the $\tilde{E}_1$ fixed point. It has two simple BPS states, the W and the instantonic quark, I_Q, which is a strip (we consider now only positive $m_0 = 1/g_0^2$). The (n_e, I) charges are $(1, 0)$ for the W and $(1/2, 1)$ for I_Q. These two have a bound state Δ with charge $(3/2, 1)$. Consider a state made out of m W's and n Δ's. We see that a bound state exists when $n > 0$. The spectrum in charge space is shown in figure 9. The number of bosonic zero modes is

$$n_{BZM} + 1 = \frac{(m+n+2)(m+n+1) - m(m+1)}{2}\,. \tag{5.4}$$

The corresponding geometry is $\mathbf{F}_1$, the blow-up of $\mathbf{P}^2$ at one point, p. I_Q is the exceptional divisor, E. Δ is the line, L. W corresponds to a line that passes through p, and belongs to the homology class $L - E$. To see that it is the correct homology

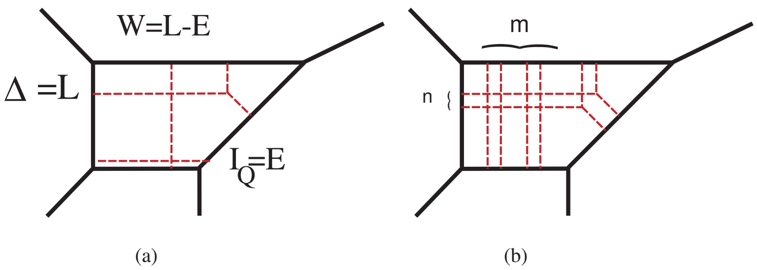

Figure 8: The $\tilde{E}_1$ theory. (a) The W, the instantonic quark I_Q and Δ. (b) A bound state of m W's and n Δ's.

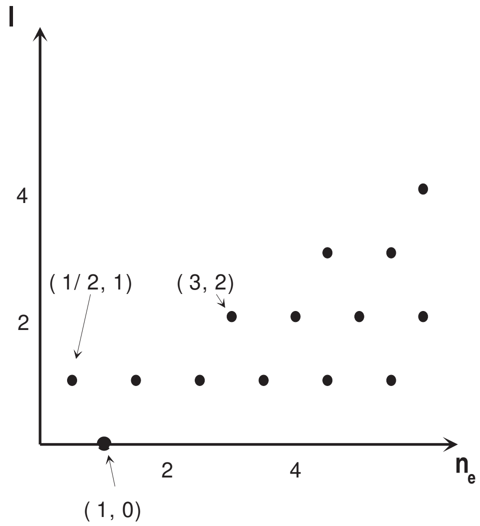

Figure 9: The sites on the $\tilde{E}_1$ charge lattice which are occupied by BPS states.

class we have to know the intersection algebra on $\mathbf{F}_1$: $L \cdot L = 1, L \cdot E = 0, E \cdot E = -1$, so indeed $(L - E) \cdot E = 1$. To identify the curve spectrum we will use the fact that the intersection number of curves (belonging to different classes) is non-negative. Consider the class $mW + n\Delta = (m + n)L - mE$. From intersecting it with E we get $m \geq 0$, while from intersecting it with $L - E$ we get $n \geq 0$. The case $n = 0$ is a curve of degree m passing m times through p and is clearly reducible. So the curve spectrum agrees. To find the dimension of moduli space we have to count curves of degree $m + n$ which pass through a specific point m times. Imposing that the first m derivatives vanish at p we indeed get $\dim_{\mathbb{C}}\mathcal{M} + 1 = [(m + n + 2)(m + n + 1) - m(m + 1)]/2$.

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